Thermal Boundary Layer Thickness for Laminar Forced Convection to Flat Plates with Uniform Heating and Uniform Wall Temperature

STUART W. CHURCHILL and HIROYUKI OZOE

School of Chemical Engineering
University of Pennsylvania, Philadelphia, Pennsylvania 19104

The ratio of the thickness of the thermal boundary layer to that of the momentum boundary layer is a useful concept in the development of approximations and solutions for heat transfer. However most discussions of this ratio are qualitative. Fisher and Knudsen (1959) computed quantitative values from the differential representations of the boundary layer for uniform wall temperature and showed that the values obtained from the integral representation of the boundary layer are not reliable.

This paper presents improved and extended values for the thermal boundary layer thickness for uniform wall temperature, a complete set of values for the thermal boundary layer thickness for uniform heat flux density, asymptotic solutions for large Pr and small Pr for both uniform wall temperature and uniform heat flux density, and a simple but accurate empirical expression for both conditions and all Pr based on these asymptotic solutions.

PRIOR RESULTS

For flow over a flat plate the component of the velocity parallel to the plate is zero at the surface and approaches the free stream velocity asymptotically with distance from the plate. The boundary layer thickness is usually defined arbitrarily as the distance at which the deviation from the free stream velocity is 1%. Using the values calculated by Howarth (1938) for the Blasius solution (1908) for laminar flow:

$$\delta = 4.91\ (x\nu/u_\infty)^{1/2} \quad (1)$$

For laminar forced convection from a flat plate a thermal boundary layer thickness Δ can similarly be defined as the distance from the plate at which $(T - T_\infty)/(T_w - T_\infty) = 0.01$.

Fisher and Knudsen (1959) computed values of Δ_T in terms of the Pohlhausen solution for a plate at uniform wall temperature using the values of Howarth for the velocity field, but a value of 4.96 rather than 4.91 as the coefficient in Equation (1). They presented a tabulation and a plot of

$$f_T\{Pr\} \equiv \Delta_T(u_\infty/x\nu)^{1/2}/2 \quad (2)$$

and also a plot of Δ_T/δ versus Pr from 10^{-3} to 10^3. They compared their results with those obtained from the integral boundary layer equations and found that the latter were in moderate error at large Pr and in great error at low Pr.

NEW RESULTS

Churchill and Ozoe (1972) recently obtained complete numerical solutions for laminar convection from a uniformly heated plate. These calculations have since been refined in order to produce accurate values of the thermal boundary layer thickness. The corresponding values of

$$f_J\{Pr\} = \Delta_J(u_\infty/x\nu)^{1/2}/2 \quad (3)$$

are given in Table 1. Values of $f_T\{Pr\}$ were also computed. They are included in Table 1 along with the values computed by Fisher and Knudsen.

The asymptotic solutions for $Pr \to 0$ and $Pr \to \infty$ for the two boundary conditions yield

$$Pr^{1/2} f_T\{0\} = \text{erfc}^{-1}\{0.01\} = 1.824 \quad (4)$$

$$Pr^{1/3} f_T\{\infty\} = \left[\frac{6}{1.328}\,\gamma^{-1}\{0.99(2.679)\}\right]^{1/3} = 2.3206 \quad (5)$$

$$Pr^{1/2} f_J\{0\} = \Phi^{-1}\{0.01\} = 1.605 \quad (6)$$

$$Pr^{1/3} f_J\{\infty\} = (1.328)^{-1/3}\,\psi^{-1}\{0.01(1.1855)\} = 2.065 \quad (7)$$

The thermal boundary layer thickness and the boundary layer ratio for both conditions are thus seen to be proportional to $Pr^{-1/2}$ for $Pr \to 0$ and to $Pr^{-1/3}$ for $Pr \to \infty$.

The computed values for finite Pr are compared with these asymptotic values in Table 2. The convergence of the computed values to the asymptotic values provides a confirmation of their accuracy. On this ground the computed values for uniform wall temperature appear to be slightly more accurate than those of Fisher and Knudsen. However, the calculations themselves suggest that the fourth significant figure in the computed values in Tables 1 and 2 may not be reliable.

The computed values of $\Delta/\delta = \Delta(u_\infty/x\nu)^{1/2}/4.91 = f\{Pr\}/2.455$ are plotted in Figure 1. The dashed lines represent the asymptotes given by Equations (4) to (7).

The following empirical equation was constructed from the limiting values according to the procedure suggested

Table 1. Computed Values for Dimensionless Thermal Boundary Layer Thickness

Pr	$f_T\{Pr\}$ This work	$f_T\{Pr\}$ Fisher & Knudsen	$f_J\{Pr\}$
0	∞	∞	∞
10^{-4}	182.9	—	160.7
10^{-3}	58.13	59.5	51.3
10^{-2}	18.83	19.4	16.57
10^{-1}	6.40	6.55	5.63
1	2.456	2.48	2.172
10	1.085	1.125	0.9648
10^2	0.500	0.505	0.4457
10^3	0.232	0.234	0.2069
∞	0	0	0

Correspondence should be addressed to S. W. Churchill. H. Ozoe is at Okayama University, Okayama, Japan.

by Churchill and Usagi (1972).

$$Pr^{1/3} f\{Pr\} = [Pr^{1/3} f\{\infty\}] \left[1 + \left(\frac{Pr^{1/2} f\{0\}}{Pr^{1/3} f\{\infty\}} \right)^5 \frac{1}{Pr^{5/6}} \right]^{1/5} \quad (8)$$

This expression holds for both boundary conditions. Exponents of 4 and 1/4 represent the computed values for $Pr < 0.1$ somewhat better than 5 and 1/5 but the latter values are best overall.

Substitution of the values for $Pr^{1/2} f\{0\}$ and $Pr^{1/2} f\{\infty\}$ from Equations (4) and (5) in Equation (8) gives

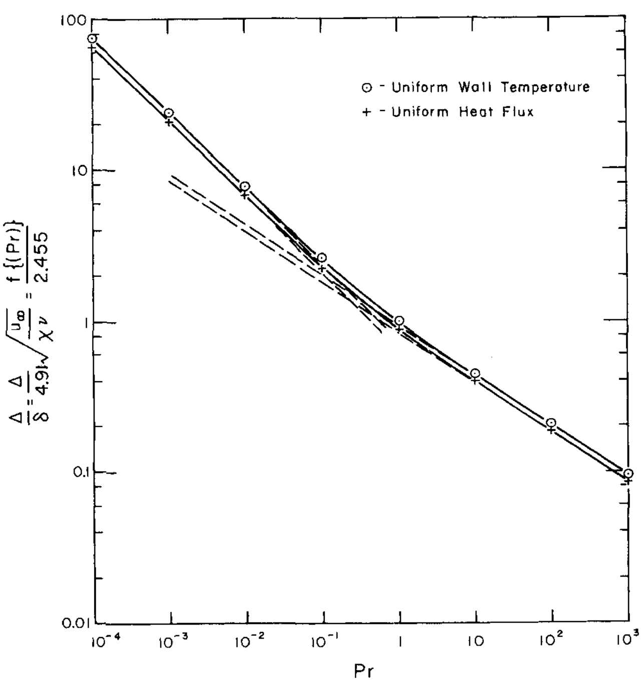
$$\frac{\Delta_T}{\delta} = \frac{0.945}{Pr^{1/3}} \left[1 + \left(\frac{0.236}{Pr} \right)^{5/6} \right]^{1/5} \quad (9)$$

Similarly, substitution from Equations (6) and (7) in (8) gives

$$\frac{\Delta_J}{\delta} = \frac{0.841}{Pr^{1/3}} \left[1 + \left(\frac{0.220}{Pr} \right)^{5/6} \right]^{1/5} \quad (10)$$

TABLE 2. COMPARISON OF COMPUTED AND ASYMPTOTIC VALUES OF $f\{Pr\}$

Pr	$Pr^{1/2} \cdot f_T\{Pr\}$ This work	$Pr^{1/2} \cdot f_T\{Pr\}$ Fisher & Knudsen	$Pr^{1/2} \cdot f_J\{Pr\}$	$Pr^{1/3} \cdot f_T\{Pr\}$ This work	$Pr^{1/3} \cdot f_T\{Pr\}$ Fisher & Knudsen	$Pr^{1/3} \cdot f_J\{Pr\}$
0	(1.824)		(1.605)			
10^{-4}	1.829	—	1.607			
10^{-3}	1.838	1.881	1.622			
10^{-2}	1.883	1.94	1.657			
10^{-1}	2.024	2.07	1.780			
1	2.456	2.48	2.172	2.456	2.48	2.172
10				2.338	2.425	2.079
10^2				2.32	2.345	2.070
10^3				2.32	2.34	2.068
∞				(2.3206)		(2.065)



The curves in Figure 1 correspond to Equations (9) and (10). The equations rather than the graph are recommended for numerical work and for subsequent derivations involving the thermal boundary layer.

The boundary layer ratio is exactly unity at $Pr = 1$ for uniform wall temperature. However, the computed values and Equation (10) reveal that this ratio is less than unity (0.885) at $Pr = 1$ for uniform heating since the boundary conditions for the momentum and energy equations are not analogous. The boundary layer ratio (and the thermal boundary layer thickness) is indeed approximately 13% greater for uniform wall temperature than for uniform heating for all Pr.

This work demonstrates the great value of asymptotic solutions even when numerical solutions are available. The asymptotic solutions presented herein reveal the exact dependence on Pr in the limits, provide the basis for construction of an empirical equation for interpolation, and provide a critical and independent test of the accuracy of the values obtained by numerical integration.

NOTATION

$f\{Pr\}$ = $\Delta(U_\infty/x\nu)^{1/2}/2$
Pr = ν/α = Prandtl number
T = temperature, K
u = velocity, m/s
x = distance from leading edge of plate, m
z = a general variable
α = thermal diffusivity, m^2/s
$\gamma(z, 1/3)$ = $\int_0^z z^{-2/3} e^{-z} dz$ = incomplete gamma function of 1/3
δ = thickness of velocity boundary layer ($u = 0.99\, u_\infty$), m
Δ = thickness of thermal boundary layer $(T - T_\infty)/(T_w - T_\infty) = 0.01$, m
ν = kinematic viscosity, m^2/s
$\Phi(z)$ $\equiv$ $e^{-z^2} - \pi^{1/2} z \operatorname{erfc} z$
$\psi(z)$ $\equiv$ solution of $\psi'' + \frac{z^2}{2}\psi' - z\psi = 0$ with $\psi = 0$ at $z = 0$ and $\psi' = -1$ at $z = 0$

Subscripts

J = uniform heat flux density
T = uniform wall temperature
w = wall
∞ = free stream

LITERATURE CITED

1. Howarth, L., "On the Solution of the Laminar Boundary Layer Equations," *Proc. Roy. Soc., Ser. A*, **164**, 547 (1938).
2. Blasius, H., "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Z. Math. Phys.*, **56**, 1 (1908).
3. Fisher, F. D., and J. G. Knudsen, "Heat Transfer during Laminar Flow Past Flat Plates: An Extension of Pohlhausen's Solution to Low- and High-Prandtl Number Fluids," *Chem. Eng. Prog. Symp. Ser. No. 29*, **55**, 209 (1959).
4. Churchill, S. W., and Hiroyuki Ozoe, "Correlations for Laminar Forced Convection in Flow over a Uniformly Heated Plate and in Developing a Fully Developed Flow in a Uniformly Heated Tube," Trans. A.S.M.E., *J. Heat Transfer* (in press).
5. Churchill, S. W., and R. Usagi, "A General Expression for the Correlation of Rates of Transfer and Other Phenomena," AIChE J., **18**, 000 (1972).

Manuscript received July 25, 1972; note accepted July 26, 1972.